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Curvature and the Eigenvalues of the Dolbeault Complex for Kaehler Manifolds

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INTRODUCTION

Let M be a compact $2m$-dimensional complex Kaehler manifold. There is a natural connection on the complex tangent bundle. We use this connection to construct forms in the DeRham cohomology to represent the Chern classes of M. These forms can be computed by functorial expressions in the first and second order derivatives of the metric tensor g. It is clear from the definition that these forms of order $2m$ vanish identically on any manifold $M = T_2 \times N_{2m-2}$ which is given the product metric. T_2 is the flat 2 torus. This property uniquely characterizes the Chern classes. We will give an analytic proof of the Riemann–Roch theorem for Kaehler manifold by using this property.

Let E be a holomorphic vector bundle with a Hermetian metric h. There is a functorial connection defined on E with connection form $\partial h \cdot h^{-1}$. This enables us to express the Chern classes of E functorially in terms of the first and second order derivatives of the metric tensor h. If E is a vector bundle over $M = T_2 \times N_{2m-2}$ which is given a product metric flat over T_2, then the classes of degree $2m$ also vanish identically.

A map P from Kaehler metrics g on M and Hermetian metrics h on E to $2m$ forms on M is given by a local formula if we can express P functorially as a polynomial in the derivatives of the metrics g and h in any coordinate system and relative to any frame for E. If A is any monomial of P, we define the order of A by counting the total number of derivatives in A. P is of order n if P is composed only of monomials of order n. Such local formulae have been discussed in the real case in [1, 3].

We can construct maps of order $2m$ by taking combinations of Chern classes on M and E. Such maps will map metrics g, h to $2m$ forms and will vanish identically on all manifolds of the form $M = T_2 \times N_{2m-2}$ where g, h are product metrics flat over T_2. In this paper, we will prove that this property uniquely characterizes these polynomials.

THEOREM 1. *Let P be any map from Kaehler metrics g on T^*M and Hermetian metrics h on E to $2m$ forms. Suppose that P is given by a local formula of order $n \leqslant 2m$ and that P vanishes identically on all product metrics over $M = T_2 \times N_{2m-2}$ which are flat over T_2. If $n < 2m$, P vanishes identically, while if $n = 2m$, P can be computed as a combination of the Chern classes of E and T^*M.*

There is a similar theorem in the real case which classifies the Pontrjagin and Euler classes of M and the Chern classes of a bundle E as maps from metrics and connections to forms. This theorem may be found in [3] or in a recent paper by Atiyah, Bott, and Patodi [1].

Such maps arise from the study of the asymptotic behavior of the eigenvalues of an elliptic operator. Let M be any compact smooth manifold and $V^\pm$ smooth vector bundles over M. Let:

$$d\colon \Gamma(V^\pm) \to \Gamma(V^\mp)$$

be a first order elliptic complex. We define the associated Laplace operators $D^\pm = (d^*d + dd^*)$ on $\Gamma(V^\pm)$. Let $\{a_j^\pm\}$ be the eigenvalues of $D^\pm$ and let $\{\theta_j^\pm\}$ be a spectral decomposition. By using the calculus of pseudodifferential operators, Seeley [5, 6] has shown that the series:

$$f(t, x, d) = \sum \exp(-ta_j^+)(\theta_j^+, \theta_j^+)(x) - \sum \exp(-ta_j^-)(\theta_j^-, \theta_j^-)(x)$$

is well defined for $\mathrm{Re}(t) > 0$ and has an asymptotic expansions as $t \to 0^+$ of the form:

$$f(t, x, d) \sim \sum_{n=0}^{\infty} B_n(x, d)\, t^{(n-m)/2}.$$

The invariants $B_n(x, d)$ are local invariants of the operator d and can be computed in any local system in terms of the derivatives of the symbol of the operator d. From the formal definition it is easily verified that

$$\int f(t, x, d) = \mathrm{index}(d)$$

and therefore

$$\int B_n(x, d) = \begin{cases} 0 & \text{for} \quad n \neq m = \dim(M), \\ \mathrm{index}(d) & \text{for} \quad n = m = \dim(M). \end{cases}$$

A proof of the Atiyah–Singer index theorem using these invariants and the classification theorem in the real case may be found in [1, 3].

Atiyah, Bott, and Patodi give a direct proof of the Riemann–Roch theorem for Kaehler manifolds by using the dirac operator and the classification theorem for the real case. In this paper, we will use Theorem 1 to prove the Riemann–Roch theorem by identifying B_{2m} for the Dolbeault complex with the Riemann–Roch invariant. Patodi (4) proved this result previously by making a direct computation. Our proof in this paper substitutes invariance theory for his computations.

If M is a complex Kaehler manifold, there is a natural orientation for M and we may identify the space of forms of degree $2m$ with the space of functions on M. The invariant $B_n(x, d)$ for the twisted Dolbeault complex is a map from metrics g, h to $2m$ forms. It is clear from the multiplicative properties of the Dolbeault complex and from certain formal properties concerning the invariants B_n that B_n is of order n in the derivatives of g, h and that $B_n(x, d)$ vanishes identically on manifolds of the form $M = T_2 \times N_{2m-2}$ where g, h are flat over T_2. Since $\int B_{2m} =$ index(d), it suffices to prove that $B_{2m}(x, d)$ is the Riemann–Roch invariant in order to prove the Riemann–Roch theorem. From Theorem 1 it follows that $B_n(x, d) = 0$ for $n < 2m$ and that $B_{2m}(x, d)$ is some functorial combination of Chern classes on M and E. Since such an index formula must be unique, this proves the Riemann-Roch theorem.

It should be noted that this local result concerning the invariants $B_n(x, d)$ is *not* true for a generic non-Kaehler manifold. It may be shown [2] that $B_n(x, d) \neq 0$ for $n \geqslant m$ and that $B_{2m}(x, d) \neq$ the Riemann–Roch invariant for $m > 1$ in the generic case. In some fashion, the Kaehler condition imposes additional relations on the metric which forces these extra terms to cancel out.

1

Let M be a complex manifold of dimension $2m$ and let z_o be a point of M. Let $Z_a = X_a + iY_a$ be a system of holomorphic coordinates centred at x_o. Define

$$d/dZ_a = (d/dX_a - id/dY_a)/2 \qquad \text{and} \qquad d/d\bar{Z}_a = (d/dX_a + id/Y_a)/2$$
$$a = 1, \dots, m$$

We extend the metric g to be bilinear on $TM \otimes C$ and define

$$g_{a\bar{b}} = G(d/dZ_a, d/d\bar{Z}_b)$$

Since g is Hermetian,

$$g_{ab} = g_{\bar{a}\bar{b}} = 0 \qquad \text{and} \qquad g_{a\bar{b}} = \bar{g}_{b\bar{a}}$$

Let E be a holomorphic vector bundle with hermetian metric h, and let $\{\zeta_q\}$ be a holomorphic frame for E. Define

$$h_{p\bar{q}} = h(\zeta_p, \zeta_q) = \bar{h}_{q\bar{p}}$$

We use two different notations to denote derivatives of the metric tensors.

(1) If $w = (w(1),..., w(m))$ is a multi-index, define $d_w = \prod_{a=1}^{m} (d/dZ_a)^{w(a)}$ and $d_{\bar{w}} = \prod_{a=1}^{m} (d/d\bar{Z}_a)^{w(a)}$. Define $g_{a\bar{b}/w\bar{w}'} = d_w d_{\bar{w}'}(g_{a\bar{b}})$ and $h_{p\bar{q}/w\bar{w}'} = d_w d_{\bar{w}'}(h_{p\bar{q}})$. Set $w! = \prod_{a=1}^{m} w(a)!$ and $|w| = \sum_{a=1}^{m} w(a)$.

(2) Let

$$g_{a\bar{b}/a_1^{v_1}\cdots\bar{b}_1^{v_1'}\cdots} = d/dZ_{a_1}^{v_1} \cdots d/d\bar{Z}_{b_1}^{v_1'} \cdots (g_{a\bar{b}})$$

and

$$h_{p\bar{q}/a_1^{v_1}\cdots\bar{b}_1^{v_1'}\cdots} = d/dZ_{a_1}^{v_1} \cdots d/d\bar{Z}_{b_1}^{v_1'} \cdots (h_{p\bar{q}}).$$

If the $v_i = 1$, we will omit them.

The Kaehler 2 form is invariantly defined by

$$\Omega = \sum g_{a\bar{b}}\, dZ_a\, d\bar{Z}_b$$

We assume g is a Kaehler metric; that is $d\Omega = 0$. This is equivalent to the following relations:

$$g_{a\bar{b}/c} = g_{c\bar{b}/a} \qquad \text{and} \qquad g_{a\bar{b}/\bar{c}} = g_{a\bar{c}/\bar{b}}\,.$$

We differentiate these relations to obtain:

$$g_{a\bar{b}/a_1\cdots\bar{b}_1\cdots} = g_{a_1\bar{b}/a\cdots\bar{b}_1\cdots} = g_{a\bar{b}_1/a_1\cdots\bar{b}\cdots}$$

We define new variables

$$g(a_o \cdots; b_o \cdots) = g_{a_o\bar{b}_o/a_1\cdots\bar{b}_1\cdots}$$

which are symmetric in the $a_o \cdots$ and $b_o \cdots$ induces. From this symmetry, we may also use the multi-index notation $g(w; w')$ to denote this variable.

Any polynomial in the derivatives of the metric can be expressed in terms of these new variables and conversely.

Let $P = P(g(w; w'), h_{p\bar{q}/ww'})$ be a polynomial in these variables. Given metrics g, h, a coordinate system Z_a and a frame ζ_p for E, we can evaluate P at z_o. We assume the coordinate system is orthonormal at z_o, i.e., that $g_{a\bar{b}}(x_o) = \delta_{ab}$, and that the frame is orthonormal at z_o, i.e., that $h_{p\bar{q}}(z_o) = \delta_{pq}$. We can further normalize the holomorphic coordinate system and reduce the structure group to the unitary group.

LEMMA 2. *Let $n \geqslant 1$. There is a holomorphic coordinate system Z_a^n and frame ζ_p^n such that in this system*

(1) $g(a; b) = \delta_{a,b}$, $g(w; b) = g(a; w) = 0$ *at* z_o *for* $1 < | w | \leqslant n$;

(2) $h_{p\bar{q}} = \delta_{p,q}$, $h_{p\bar{q}/w} = h_{p\bar{q}/\bar{w}} = 0$ *at* z_o *for* $1 \leqslant | w | \leqslant n - 1$.

The proof is by induction. The case $n = 1$ has been discussed previously. We now assume $n > 1$ and $Z_a^{(n-1)} = Z_a$ is the coordinate system for $n = n - 1$. Let Z_a' be the holomorphic coordinate system given by

$$Z_a = Z_a' + \sum_{|w|=n} C_{a,w} Z'^w \qquad \text{for some constants } C_{a,w}$$

Since $Z_a' = Z_a + O(Z^n)$, it follows that $g(a; w)$ is unchanged if $| w | < n$ and hence vanishes by induction at z_o. If $1 \leqslant a \leqslant m$ and w is a multi-index, define $w_a = \phi$ if $w(a) = 0$, and otherwise define

$$w_a(b) = \begin{cases} w(b) & \text{for} \quad b \neq a, \\ w(a) - 1 & \text{for} \quad b = a. \end{cases}$$

With this notation,

$$d/dZ_a' = \sum dZ_b/dZ_a' \cdot d/dZ_b = \sum C_{b,w} Z'^{w_a} d/dZ_b w(a) + d/dZ_a$$

Since $g_{a\bar{b}} = \delta_{ab} + O(Z)$ and $Z = Z' + O(Z^n)$,

$$g'_{a\bar{b}} = g_{a\bar{b}} + C_{b,w} w(a)\, Z^{w_a} + \text{terms in } \bar{Z} + \text{terms } O(Z^n)$$

Since $d/dZ'^{w_a} = d/dZ^{w_a} + O(Z^{n-1})$,

$$g'(w; b) = g(w; b) + w! C_{b,w} + O(Z)$$

We define $C_{b,w} = -g(w; b)(z_o)/w!$ to satisfy $g'(w; b)(z_o) = 0$ for

$|w| = n$. Since $g'(a; w)(z_o) = \bar{g}'(w; a)(z_o)$, this also implies $g'(a; w) = 0$ for $|w| = n$.

We construct a new frame to satisfy condition (2) in a similar fashion. Let ζ_p be the frame constructed for $n = n - 1$ and define a new holomorphic frame

$$\zeta_p' = \zeta_p + f_{p,q}(z)\,\zeta_q \qquad \text{where } f_{pq} \text{ is analytic and } O(Z^{n-1}).$$

The assumption $h_{p\bar{q}} = \delta_{pq} + O(Z)$ implies that

$$h'_{p\bar{q}} = h_{p\bar{q}} + f_{pq} + \text{terms in } \bar{Z} + \text{terms } O(Z^n)$$

Consequently

$$h'_{p\bar{q}/w} = h_{p\bar{q}/w} + d_w(f_{pq}) + O(Z)$$

By induction, $h_{p\bar{q}/w}(z_o) = 0$ for $|w| \leqslant n - 2$ and we define f_{pq} such that $d_w(f_{pq}) = -h_{p\bar{q}/w}(z_o)$ for $|w| = n - 1$. Since $h_{p\bar{q}/\bar{w}} = \bar{h}_{p\bar{q}/w}(z_o) = 0$, this proves Lemma 2.

This lemma enables us to normalize the coordinate system and frame up to unitary transformations and holomorphic transformations of arbitrarily high order.

2

We make the following definitions.

$$A = g(w_1\,;\,w_1') \cdots g(w_j\,;\,w_j')\,h_{p_{j+1},\bar{q}_{j+1}/w_{j+1},w'_{j+1}} \cdots h_{p_k\bar{q}_k/w_kw_k'}$$

is any monomial in the derivatives of the metric tensor. Then

$$\begin{aligned}
\deg_a(A) &= w_1(a) + \cdots + w_k(a),\\
\deg_{\bar{a}}(A) &= w_1'(a) + \cdots + w_k'(a),\\
\operatorname{ord}(A) &= |w_1| + \cdots + |w_k'| - 2j,\\
L(A) &= k, \qquad c_P(A) = \text{the coefficient of } A \text{ in the polynomial } P.
\end{aligned}$$

In the definition of $\operatorname{ord}(A)$, we subtract $2 \cdot j$ to account for the fact that the variable $g(w; w')$ is of order $|w| + |w'| - 2$ in the derivatives of the metric.

Let P be any polynomial which is homogeneous of order n in these variables. P is *invariant* if the value of P depends only on the metrics g, h and is independent of the frame and coordinate system. Let P be invariant and suppose that P vanishes identically on all product manifolds $M = T_2 \times N_{2m-2}$ where g, h are flat along T_2. It is clear that we can construct germs of metrics g, h such that $g(w; w')(z_o) = c_{w,w'}$ is any arbitrary constant and $h_{p\bar{q}/w\bar{w}'} = d^{pq}_{w,w'}$ is any arbitrary constant provided that $c_{w,w'} = \bar{c}_{w,w'}$ and $d^{pq}_{w,w'} = \bar{d}^{qp}_{w'w}$. Consequently, P vanishes for all g, h if and only if P is the zero polynomial. If we restrict P to $M = T_2 \times N_{2m-2}$ we introduce the relation that $A = 0$ if $\deg_1(A) + \deg_{\bar{1}}(A) > 0$. This implies that P is in the ideal generated by this relation.

We obtain further information about P by exploiting the invariance of P. We are free to use a unitary change of coordinates and keep the coordinate system normalized. Let $Z' = Z'(Z)$ be a unitary coordinate change such that:

$$
\begin{aligned}
&d/dZ_a' = r\,d/dZ_a + s\,d/dZ_b && d/d\bar{Z}_a' = \bar{r}\,d/d\bar{Z}_a + \bar{s}\,d/d\bar{Z}_b && r\bar{r} + s\bar{s} = 1\\
&d/dZ_b' = -\bar{s}\,d/dZ_a + \bar{r}\,d/dZ_b && d/d\bar{Z}_b' = -s\,d/d\bar{Z}_a + r\,d/d\bar{Z}_b &&\\
&d/dZ_c' = d/dZ_c && d/d\bar{Z}_c = d/d\bar{Z}_c && c \neq a, b
\end{aligned}
$$

First we use the case that $a = b$ and we have a coordinate change

$$d/dZ_a' = e^{i\phi} d/dZ_a \qquad d/d\bar{Z}_a' = e^{-i\phi} d/d\bar{Z}_a\,.$$

If A is any monomial of P, when we compute A' in the new coordinate system, it is clear that $A' = e^{(\deg_a - \deg_{\bar{a}})\phi} A$. Since P is invariant, the *form* of P must be invariant under this transformation. This implies that $P' = P$. Since ϕ is arbitrary, this implies $\deg_a(A) = \deg_{\bar{a}}(A)$ for all a, A.

By assumption, $\deg_1(A) + \deg_{\bar{1}}(A) > 0$. If $a = 1$, $r = 0$, $s = 1$ this unitary rotation permutes the first and b-th indices and hence $\deg_b(A) + \deg_{\bar{b}}(A) > 0$ for all b. This implies that $\deg_b(A) = \deg_{\bar{b}}(A) > 0$ for all b, A. The following lemma proves that P vanishes identically if $\operatorname{ord}(P) = n < 2m$.

LEMMA 3. *Let P be as above, and let A be some monomial of P with $L(A) = k \leqslant m$ then*:

(1) *there is some monomial A_o of P of the form*

$$A_o = g(11 \cdots 1; w_1') \cdots h_{p_k\bar{q}_k/k\cdots k,\bar{w}_k'}\,;$$

(2) *P vanishes if* $\operatorname{ord}(A) = \operatorname{ord}(P) < 2m$;

(3) *if* $\operatorname{ord}(P) = 2m$, *then* $L(A) = m$ *for every* A *a monomial of* P, *and* A *consists only of second-order terms. Thus* $A = g(11; i_1 i_1') \cdots h_{p_m \bar{q}_m / m i_m'}$.

Proof. We deduce (2) and (3) from (1) as follows: given such an A_o, it follows that $\deg_m(A_o) = 0$ for $m > k$ and hence $k \geqslant m$. We have restricted ourselves to coordinate systems normalized by Lemma 2, and hence we consider only variables of order $\geqslant 2$ which involve both d/dZ and $d/d\bar{Z}$ derivatives. Consequently, $\operatorname{ord}(P) \geqslant 2L(A) \geqslant 2m$. This implies P vanishes if $\operatorname{ord}(P) < 2m$. If $\operatorname{ord}(P) = 2m$, all the inequalities must be equalities and P involves only second order derivatives.

The proof of (1) is by induction. We construct a monomial A_o' such that

$$A_o' = g(1 \cdots 1; w_1') \cdots g(j \cdots j; w_j')\, h_{p_{j+1}, \bar{q}_{j+1}/w_{j+1}, w_{j+1}'} \cdots .$$

Choose $a \leqslant j + 1$ so that for some multi-indices $w_1' \cdots w_{a-1}'$ some monomial of P is divisible by $g(1 \cdots 1; w_1') \cdots g(a - 1, \ldots, a - 1; w_{a-1}')$. If $a = j + 1$, the assertion is proved so we assume $a \leqslant j$. Consequently, some monomial of P is of the form

$$A_1 = g(1 \cdots 1; w_1') \cdots g(a - 1, \ldots, a - 1; w_{a-1}')\, g(a^v b^{v'} \cdots; w_a') \cdots .$$

Choose A_1 so that v is maximal. Since a was maximal, $g(a^v \cdots; w_a')$ must involve at least one other index and hence $v' > 0$. We obtain a contradiction as follows: let U denote the unitary coordinate rotation in the a, b axes discussed above. In order to compute P' in the new coordinate system, we replace each index by:

$$a \to ra + sb \qquad \bar{a} \to \overline{ra} + \overline{sb} \qquad b \to -\bar{s}a + \bar{r}b \qquad \bar{b} \to -s\bar{a} + r\bar{b}.$$

We expand the resulting expression and recombine terms. Since P is invariant, the form of P is unchanged by this procedure. Let A_2 be the monomial which is obtained from A_1 by changing a single $b \to a$ index:

$$A_2 = g(1 \cdots 1; w_1') \cdots g(a - 1, \ldots, a - 1; w_{a-1}')\, g(a^{v+1} b^{v-1} \cdots; w_a') \cdots .$$

In the new coordinate system:

$$P' = f(s, \bar{s}, r, \bar{r})\, A_2 + \text{other terms},$$

where f is homogeneous of degree $2^*(\deg_a(A_1) + \deg_b(A_1)) = 2$ in the s, r variables. Since $\deg_a(A_2) + \deg_{\bar{a}}(A_2) = 2^* \deg_a(A_1) + 1$ is odd, this implies A_2 is not a monomial of $P = P'$. Thus $f(s, \bar{s}, r, \bar{r})$ vanishes for all (s, r) such that $s\bar{s} + r\bar{r} = 1$. Since f is homogeneous, this implies $f(s, \bar{s}, r, \bar{r})$ vanishes for all (s, r) and thus $f = 0$.

We let r be real and consider the set of all monomials B which give rise to a term $c_B r^{t-1}\bar{s}A_2 + \cdots$ in the new coordinate system. Since the exponent of $\bar{s}$ is 1, A_2 is obtained from B by changing a single $b \to a$ or $\bar{a} \to \bar{b}$ index. In the latter case, B must be of the form

$$B = g(1 \cdots 1; w_1'') \cdots g(a^{v+1}b^{v-1} \cdots; w_a'') \cdots .$$

This contradicts the maximality of a, v and thus B changes to A_2 by changing a single index $b \to a$. If $B \neq A_1$, then B is of the form

$$B = g(1 \cdots 1; w_1') \cdots g(a^{v+1}b^{v-1} \cdots) \cdots ,$$

which again contradicts the choice of a, v. This implies $B = A_1$. Since $A_1' = v'r^{t-1}\bar{s}A_2 +$ other terms, this implies the coefficient of $r^{t-1}\bar{s}$ in f is nonzero. This contradiction establishes the existence of a monomial A_o' of the form

$$A_o' = g(1 \cdots 1; w_1') \cdots g(j \cdots j; w_j')\, h_{p_{j+1},\bar{q}_{j+1}/w_{j+1},\bar{w}'_{j+1}} \cdots h_{p_k q_k/w_k \bar{w}_k'} .$$

We use this method again to complete the proof. Choose $a \leqslant k + 1$ maximal so that

$$g(1 \cdots 1; w_1') \cdots h_{p_{a-1},\bar{q}_{a-1}/a-1,\ldots,a-1,\bar{w}'_{a-1}} = C$$

divides some monomial of P. If $a = k + 1$, the lemma is proved so we suppose $a \leqslant k$. Choose v maximal such that $C'h_{p_a\bar{q}_a/a^v b^{v'}}\ldots$ divides some monomial of P where C' is of the form above. By using exactly the same argument, this implies $v' = 0$ and completes the proof.

We will use the following lemma to separate variables and to reduce the classification problem under consideration to the corresponding problem for invariant polynomials in the metrics separately. Since $P = 0$ unless $\operatorname{ord}(P) = 2m$, we assume this from now on.

LEMMA 4. *Let P be as above. Then some monomial of P is of the form*

$$A = g(11; i_1 i_1) \cdots g(jj; i_j i_j)\, h_{p_{j+1}\bar{q}_{j+1}/j+1,\bar{i}_{j+1}} \cdots h_{p_m \bar{q}_m/m\bar{i}_m} .$$

Proof. From Lemma 3 it follows that some monomial A_o is of the form:

$$A_o = g(11; i_1 i_1') \cdots g(jj; i_j i_j') \cdots h_{p_m q_m/m\bar{i}_m'}$$

From this representation it follows that

$$\deg_a(A_o) = \deg_{\bar a}(A_o) = \begin{cases} 2 & \text{if } a \leqslant j, \\ 1 & \text{if } a > j, \end{cases}$$

Choose A_0 so that the number of times that $i_a = i_a'$ is maximal. If this can be done for all $a \leqslant j$, the lemma is proved so assume A_0 divisible by $g(ii; ab)$ for $a \neq b$.

If A is any monomial of P, let $c_p(A)$ denote the coefficient of A in P. We use the symbol "*" to denote terms not of interest and the symbol "$\cdots$" to indicate missing factors in a monomial. Express $A_o = g(*; ab) \cdots$ and assume first that $a, b > j$. Let A_1 denote the corresponding monomial $g(*; aa) \cdots$ and consider the unitary rotation of the a, b axes. We compute that the coefficient of $r^3 s A_1$ in the polynomial P' must be zero. If B is any monomial which gives rise to such a term in the new coordinate system, A_1 is obtained from B by changing a single $\bar b \to \bar a$ or $a \to b$ index. In the latter case, $\deg_b(B) = 0$ which is impossible. In the former case, since $\deg_{\bar a}(A_1) = 2$, the only place in which $\bar a$ occurs is in the variable $g(*; aa)$ and hence $B = A_o$. This implies the coefficient of $r^3 s A_1$ is nonzero and shows not both a and $b > j$.

Next we suppose that $a \leqslant j$ but that $b > j$. In this case, $\deg_a(A_o) = 2$ and $\deg_b(A_o) = 1$. We express

$$A_o = g(*; ab)\, g(*; ax) \cdots \qquad \text{or} \quad g(*; ab)\, h_{p\bar q/x\bar a} \cdots .$$

We define

$$A_{01} = g(*; aa)\, g(*; ax) \cdots \qquad \text{or} \quad g(*; aa)\, h_{p\bar q/x\bar a} \cdots ,$$

$$A_1 = g(*; aa)\, g(*; bx) \cdots \qquad \text{or} \quad g(*; aa)\, h_{p\bar q/x\bar b} \cdots .$$

The coefficient of $r^5 s A_{01}$ in P' is zero. Let B be any monomial which gives rise to such a term in the new coordinate system. B changes to A_{01} by altering an $\bar b \to \bar a$ or $a \to b$ index. In the latter case, $\deg_b(B) = 0$ which is false. The only two possibilities in the former case are A_o, A_1. Since A_1 is in the same form as is A_0 with one more double index, by the maximality of A_0 this implies A_1 is not a monomial of P. Consequently since $A_o' = -sr^5 A_{01} + \cdots$, this implies the coefficient of $r^5 s A_{01}$ in P' is nonzero. This contradiction implies $a, b \leqslant j$.

If we factor $A_o = B_o C_o$ where B_o consists of the $g(w; w')$ variables of A_o and C_o of the $h_{p\bar q/w\bar w'}$ variables, then we show as follows that $\deg_{\bar a}(C_o) = 0$ for $a \leqslant j$ and $\deg_{\bar a}(B_o) = 0$ for $a > j$. This implies the monomial splits into two parts which do not interact. We already showed that $\deg_{\bar{a}}(B_o) = 0$ for $a > j$. Consequently $m - j = \sum \deg_{\bar{a}} C_o = L(C_o) = \sum_{a>j} \deg_{\bar{a}}(A_o) = \sum_{a>j} \deg_{\bar{a}} C_o$. This implies $\deg_{\bar{a}}(C_o) = 0$ for $a < j$.

We may therefore express A_o in form

$$A_o = g(bb;\, *)\, g(*;\, ab)\, g(*;\, bx) \cdots$$

This is the generic case. The case in which $A_o = g(bb;\, ab)\, g(*;\, bx) \cdots$ can be treated in a similar fashion. We assume now that $a \neq x$ and therefore

$$A_o = g(bb;\, *)\, g(*;\, ab)\, g(*;\, xb)\, g(*;\, ya) \cdots .$$

We define

$$A_1 = g(bb;\, *)\, g(*;\, aa)\, g(*;\, xb)\, g(*;\, yb) \cdots ,$$

$$A_2 = g(ab;\, *)\, g(*;\, aa)\, g(*;\, xb)\, g(*;\, ya) \cdots ,$$

$$A_{012} = g(bb;\, *)\, g(*;\, aa)\, g(*;\, xb)\, g(*;\, ya) \cdots .$$

The coefficient of A_{012} in P' must vanish identically. We consider those monomials B which give rise to an $r^7 s A_{012}$ term. Since B changes to A_{012} by changing an $a \to b$ or $\bar{b} \to \bar{a}$ index, it is clear that $B = A_0$, A_1, A_2. By the maximality of A_o, A_1 cannot be a monomial of P. Since $A_2' = sr^7 A_{012} + \cdots$ and $A_o' = -sr^7 A_{012}$, this implies $c_P(A_0) = c_P(A_2) \neq 0$.

By assumption, $x \neq a$ and thus we can define

$$A_2 = g(ab;\, *)\, g(*;\, aa)\, g(*;\, xb)\, g(*;\, ya)\, g(*;\, xz) \cdots,$$

$$A_3 = g(ab;\, *)\, g(*;\, aa)\, g(*;\, xx)\, g(*;\, ya)\, g(*;\, bz) \cdots,$$

$$A_{23} = g(ab;\, *)\, g(*;\, aa)\, g(*;\, xx)\, g(*;\, ya)\, g(*;\, xz) \cdots .$$

We rotate the x and b coordinates. As before, the coefficient of A_{23} in P' must vanish. If B changes to A_{23} by altering an $x \to b$ index, then $\deg_b(B) = 0$ which is impossible. Consequently the only terms to be considered are A_2 or A_3. Since $A_2' = -sr^5 A_{23} + \cdots$ and $A_3' = -sr^5 A_{23} + \cdots$, this implies that $c_P(A_2) = -c_P(A_3) \neq 0$. We reverse the previous argument to show that this implies

$$A_4 = g(bb;\, *)\, g(*;\, ab)\, g(*;\, xx)\, g(*;\, ya)\, g(*;\, bz) \cdots$$

is a monomial of P. A_4 is of the same form as A_0 and contains one more double index in the second argument. This contradicts the maximality of A_o and implies $a = x$.

Thus A_o is of the form

$$A_o = g(bb;\ ^*)\, g(^*;\ ab)\, g(^*;\ ab) \cdots .$$

We define

$$A_1 = g(ab;\ ^*)\, g(^*;\ aa)\, g(^*;\ ab) \cdots,$$

$$A_2 = g(ab;\ ^*)\, g(^*;\ ab)\, g(^*;\ aa) \cdots,$$

and argue as before to show that $c_P(A_1) = c_P(A_o) = c_P(A_2)$. We define

$$A_{12} = g(ab;\ ^*)\, g(^*;\ aa)\, g(^*;\ aa).$$

Since A_{12} is not a monomial of P', this implies $c_P(A_1) + c_P(A_2) = 0$ and hence $c_P(A_o) = 0$. This contradicts the assumption that the monomial A_o was in fact a monomial of P and proves the lemma.

3

We introduce the following notation:

$\mathcal{Q}_m$ is the ring generated by the $h_{pq/w\bar{w}'}$ variables on a complex $2m$-dimensional manifold;

$\mathcal{Q}_{m,j}$ is the subring of $\mathcal{Q}_m$ generated by the variables $h_{pq/w\bar{w}'}$ with $w(a) = w'(a) = 0$ for $a \leqslant j$;

$\mathcal{P}_m$ is the ring generated by the $g(w;\ w')$ variables on a complex $2m$-dimensional manifold;

$\mathcal{P}_{m,j}$ is the subring of $\mathcal{P}_m$ generated by the $g(w;\ w')$ variables with $w(a) = w'(a) = 0$ for $a > j$.

There is a natural isomorphism from $\mathcal{P}_{m,j} \to \mathcal{P}_j$ and $\mathcal{Q}_{m,j} \to \mathcal{Q}_{m-j}$ which is defined by renumbering the indices which refer to the coordinate system.

We have been considering invariant elements of the algebra $\mathcal{P}_m \otimes \mathcal{Q}_m$. We define a map F_j: $\mathcal{P}_m \otimes \mathcal{Q}_m \to \mathcal{P}_{m,j} \otimes \mathcal{Q}_{m,j} = \mathcal{P}_j \otimes \mathcal{Q}_{m-j}$ by:

$$F_j(g(w;\ w')) = \begin{cases} 0 & \text{if} \quad w(a) + w'(a) \neq 0 \text{ some } a > j, \\ g(w;\ w') \otimes 1 & \text{otherwise;} \end{cases}$$

$$F_j(h_{pq/w\bar{w}'}) = \begin{cases} 0 & \text{if} \quad w(a) + w'(a) \neq 0 \text{ some } a \leqslant j \\ 1 \otimes h_{pq/w\bar{w}'} & \text{otherwise.} \end{cases}$$

Let $\mathscr{P}_m{}^n$, $\mathscr{Q}_m{}^n$, and $(\mathscr{P}_m \otimes \mathscr{Q}_m)^n$ denote the vector space of invariant polynomials of order n which vanish on product manifolds $M = T_2 \times N_{2m-2}$ with metrics flat in the T_2 direction. The following lemma separates variables and is analogous to the decomposition of P into products of Chern classes on M and on E.

LEMMA 5.

(1) F_j: $(\mathscr{P}_m \otimes \mathscr{Q}_m)^{2m} \rightarrow \mathscr{P}_j^{2j} \otimes \mathscr{Q}_{m-j}^{2m-2j}$

(2) $\oplus_{j=0}^m F_j$: $(\mathscr{P}_m \otimes \mathscr{Q}_m)^{2m} \rightarrow \oplus_{j=0}^m (\mathscr{P}_j^{2j} \otimes \mathscr{Q}_{m-j}^{2m-2j})$ *is injective*

(3) $\dim(\mathscr{P}_m \otimes \mathscr{Q}_m)^{2m} \leqslant \sum_{j=0}^m \dim(\mathscr{P}_j^{2j}) \dim(\mathscr{Q}_{m-j}^{2m-2j})$

Proof. Let $P \in (\mathscr{P}_m \otimes \mathscr{Q}_m)^{2m}$ and let Q_i be any basis for the algebra $\mathscr{Q}_{m,k}$. We decompose $F_jP = \sum P_i \otimes Q_i$. If we make a coordinate change in the first k variables, the Q_i and P are unchanged. This implies that the P_i are invariant under such coordinate transformations and it is easily checked that the P_i vanish on product manifolds of the form $M = T_2 \times N_{2m-2}$. Consequently, the $P_i \in \oplus_n \mathscr{P}_j{}^n$. We let P_i' denote a basis for this subspace and decompose $F_jP = \sum P_i' \otimes Q_i'$. If we perform a linear change in the fibre of E or a coordinate change in the last $m - k$ variables, the P_i' and P are unchanged and thus the Q_i' must be invariant. This implies $Q_i' \in \mathscr{Q}_{m-j}^n$ for some n. We have shown that $P_i' = 0$ if $\operatorname{ord}(P_i') < 2j$ and $Q_i' = 0$ if $\operatorname{ord}(Q_i') < 2m - 2j$. Since $\operatorname{ord}(P_i') + \operatorname{ord}(Q_i') = 2m$, this implies $\operatorname{ord}(P_i') = 2j$ and $\operatorname{ord}(Q_i') = 2m - 2j$ and hence $P_i' \in \mathscr{P}_j^{2j}$ and $Q_i' \in \mathscr{Q}_{m-j}^{2m-2j}$ as claimed.

We apply Lemma 4 to prove $\oplus F_j$ are injective. Let $P \neq 0$ belong to $(\mathscr{P}_m \otimes \mathscr{Q}_m)^{2m}$ and construct a monomial A_o of P of the form

$$A_o = B_oC_o \qquad \text{for} \quad B_o = g(11; i_1i_1) \cdots g(jj; i_ji_j)$$

$$C_o = h_{p_{j+1}, \bar{q}_{j+1}/j+1, i_{j+1}} \cdots h_{p_m\bar{q}_m/mi_m}$$

We have already show that $\deg_a(B_o) = \deg_{\bar{a}}(B_o) = 0$ for $a > j$ and that $\deg_a(C_o) = \deg_{\bar{a}}(C_o) = 0$ for $a \leqslant j$. This implies $F_j(A_o) = A_o$. Since F_jP is computed by discarding certain monomials of P, this implies $F_j(P) \neq 0$ for some j and thus $\oplus F_j$ is injective. Proposition (3) follows from (2) by a standard argument concerning the dimension of two vector spaces.

Let E be an r dimensional vector bundle, let $\pi_r(m)$ denote the number of partitions of m into at most r elements, and let $\pi(m) = \pi_m(m)$. We will prove that $\dim(\mathscr{P}_j^{2j}) \leqslant \pi(j)$ and that $\dim(Q_j^{2j}) \leqslant \pi_r(j)$. This will show that $\dim(\mathscr{P}_m \otimes \mathscr{Q}_m)^{2m} \leqslant \sum_{j=0}^{m} \pi_r(j)\, \pi(m-j)$. We can construct exactly $\sum_{j=0}^{m} \pi_r(j)\, \pi(m-j)$ linearly independent elements of $(\mathscr{P}_m \otimes \mathscr{Q}_m)^{2m}$ by taking all possible combinations of Chern classes of T^*M and Chern classes of E. This implies every such invariant polynomial must be a combination of Chern classes of E and T^*M and completes the proof of the classification theorem.

We let σ be a permutation of the integers $1,\ldots, m$ and define $A_\sigma = g(11; \sigma(1)\, \sigma(1)) \cdots g(mm; \sigma(m)\, \sigma(m))$. Let L_σ be the linear functional $L_\sigma(P) = c_P(A_\sigma)$. From Lemma 4 it follows that $P \neq 0$ implies that $L_\sigma(P) \neq 0$ for some σ. Thus $\dim(\mathscr{P}_m^{2m}) \leqslant$ the number of linear functionals L_σ. Conjugation of σ by any other permutation is equivalent to renaming the coordinate axes. Since P is invariant, this implies that L_σ depends only on the conjugacy class of σ. The conjugacy class of σ depends only on the length of the cycles of σ. Consequently, there are $\pi(m)$ such linear functionals L_σ and $\dim(\mathscr{P}_m^{2m}) \leqslant \pi(m)$.

In order to discuss the algebra $\mathscr{Q}_m^{2m}$, it is convenient to introduce the curvature tensor of the induced connection. We define:

$$w_{p\bar{q}} = \sum h_{p\bar{q}/a\bar{b}}\, dZ_a\, d\bar{Z}_b$$

If $P(w_{p\bar{q}})$ is homogeneous of order m, then $P(w_{p\bar{q}})$ is a $2m$ form. P is invariant if $P(w_{p\bar{q}})$ is invariant under linear changes of frame. It is automatically invariant under changes of coordinate system by the definition of the $w_{p\bar{q}}$ and by the fact that we have restricted our admissible changes of coordinates to unitary transformations. Let $\tilde{\mathscr{Q}}_m^{2m}$ denote the algebra of all such invariant polynomials. There is a natural map from $\tilde{\mathscr{Q}}_m^{2m} \to \mathscr{Q}_m^{2m}$ since such polynomials P automatically vanish on product manifolds of the form $M = T_2 \times N_{2m-2}$ on metrics flat in the T_2 direction. We show this map is surjective which implies $\dim(\mathscr{Q}_m^{2m}) \leqslant \dim(\tilde{\mathscr{Q}}_m^{2m})$.

Let $P \in \mathscr{Q}_m^{2m}$. P consists only of monomials of the form:

$$A_0 = h_{p_1\bar{q}_1/i_1\bar{i}_1'} h_{p_2\bar{q}_2/i_2\bar{i}_2'} \cdots h_{p_m q_m/i_m\bar{i}_m'}$$

where $i_1,\ldots, i_m$ and $i_1',\ldots, i_m'$ are all distinct. We define

$$A_1 = h_{p_1\bar{q}_1/i_1\bar{i}_2'} h_{p_2\bar{q}_2/i_2\bar{i}_1'} \cdots h_{p_m\bar{q}_m/i_m i_m'} \qquad (\text{switch } i_1', i_2'),$$

$$A_2 = h_{p_1\bar{q}_1/i_2\bar{i}_1'} h_{p_2\bar{q}_2/i_1\bar{i}_2'} \cdots h_{p_m\bar{q}_m/i_m\bar{i}_m'} \qquad (\text{switch } i_1, i_2),$$

then P is in the image of $\tilde{\mathscr{Q}}_m^{2m}$ iff $c_P(A_o) = -c_P(A_1) = -c_P(A_2)$ for all possible such switches. We show $c_P(A_0) + c_P(A_1) = 0$, the other case is similar.

Let

$$A_{01} = h_{p_1\bar{q}_1/i_1\bar{i}_2'}h_{p_2q_2/i_2\bar{i}_2'} \cdots .$$

We consider those monomials B of P which give rise to an r^3sA_{01} term when a unitary rotation in the i_1', i_2' axes is performed. B changes to A_{01} by rotating $\bar{i}_1' \to \bar{i}_2'$ or $i_2' \to i_1'$. This latter case is impossible since it implies $\deg_{\bar{i}_1'} B = 0$. Thus $B = A_0$ or A_1 and $c_p(A_o) + c_P(A_1) = 0$.

Thus $\dim(\tilde{\mathscr{Q}}_m^{2m}) \geqslant \dim(\mathscr{Q}_m^{2m})$. For the algebra $\tilde{\mathscr{Q}}^{2m}$, invariance under coordinate changes is automatic and therefore we only consider invariance under change of frame. It is well known that if we make a change of frame $s' = As$, then the curvature matrix changes by $w' = AwA^{-1}$. Thus $P \in \tilde{\mathscr{Q}}_m^{2m}$ is a polynomial of order m defined on $r \times r$ matrices which is invariant under conjugation. It is classical fact that there are exactly $\pi_r(m)$ such functions which completes the proof of the classification theorem. Q.E.D.

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